

MATHEMATICS

Chapter 5: COMPLEX NUMBERS & QUADRATIC EQUATIONS



COMPLEX NUMBERS & QUADRATIC EQUATIONS

Some Important Results

1. Solution of $x^2 + 1 = 0$ with the property $i^2 = -1$ is called the imaginary unit.

2. Square root of a negative real number is called an imaginary number.

3. If a and b are positive real numbers, then $\sqrt{-a} \times \sqrt{-b} = -\sqrt{ab}$

4. If a is a positive real number, then we have $\sqrt{-a} = i\sqrt{a}$.

5. Powers of i

$$i = \sqrt{-1};$$

$$i^2 = -1;$$

$$i^3 = -i$$

$$i^4 = 1$$

6. If $n > 4$, then $i^n = \frac{1}{i^k} = \frac{1}{i^k}$ where k is the remainder when n is divided by 4.

7. We have $i^0 = 1$.

8. A number in the form $a + ib$, where a and b are real numbers, is said to be a complex number.

9. In complex number $z = a + ib$, a is the real part, denoted by $\text{Re } z$ and b is the imaginary part denoted by $\text{Im } z$ of the complex number z .

10. $\sqrt{-1} = i$ is called *iota*, which is a complex number.

11. The modulus of a complex number $z = a + ib$ denoted by $|z|$ is defined to be a non-negative real number $\sqrt{a^2 + b^2}$, i.e. $|z| = \sqrt{a^2 + b^2}$.

12. For any non-zero complex number $z = a + ib$ ($a \neq 0$, $b \neq 0$), there exists a complex number $\frac{a}{a^2 + b^2} + i \frac{(-b)}{a^2 + b^2}$, denoted by $\frac{1}{z}$ or z^{-1} , called the multiplicative inverse of z such that

$$(a + ib) \times \left(\frac{a}{a^2 + b^2} + i \frac{(-b)}{a^2 + b^2} \right) = 1 + i0 = 1.$$

13. Conjugate of a complex number $z = a + ib$, denoted as $\bar{z}$, is the complex number $a - ib$.

14. The number $z = r(\cos \theta + i \sin \theta)$ is the polar form of the complex number $z = a + ib$.

Here $r = \sqrt{a^2 + b^2}$ is called the modulus of z $\theta = \tan^{-1} \left(\frac{b}{a} \right)$ and is called the argument or amplitude of z , which is denoted by $\arg z$.

15. The value of θ such that $-\pi < \theta \leq \pi$ called principal argument of z .
16. The Eulerian form of z is $z = re^{i\theta}$, where $e = \cos\theta + i\sin\theta$
17. The plane having a complex number assigned to each of its points is called the Complex plane or Argand plane.
18. Let $a_0, a_1, a_2, \dots$ be real numbers and x is a real variable. Then, the real polynomial of a real variable with real coefficients is given as
- $$f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$$
19. Let $a_0, a_1, a_2, \dots$ be complex numbers and x is a complex variable. Then, the real polynomial of a complex variable with complex coefficients is given as
- $$f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$$
20. A polynomial $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ is a polynomial of degree n .
21. Polynomial of second degree is called a quadratic polynomial.
22. Polynomials of degree 3 and 4 are known as cubic and biquadratic polynomials.
23. If $f(x)$ is a polynomial, then $f(x) = 0$ is called a polynomial equation.
24. If $f(x)$ is a quadratic polynomial, then $f(x) = 0$ is called a quadratic equation.

26. The general form of a quadratic equation is $ax^2 + bx + c = 0$, $a \neq 0$.
27. The values of the variable satisfying a given equation are called its roots.
28. A quadratic equation cannot have more than two roots.
29. Fundamental Theorem of Algebra states that 'A polynomial equation of degree n has n roots.'

Top Concepts

- Addition of two complex numbers:** If $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers, then the sum
 $z_1 + z_2 = (a + c) + i(b + d)$.
- Sum of two complex numbers is also a complex number. This is known as the closure property.
- The addition of complex numbers satisfy the following properties:
 - Addition of complex numbers satisfies the commutative law. For any two complex numbers z_1 and z_2 , $z_1 + z_2 = z_2 + z_1$.
 - Addition of complex numbers satisfies associative law for any three complex numbers z_1, z_2, z_3 , $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$.
 - There exists a complex number $0 + i0$ or 0 , called the additive identity or the zero complex number, such that for every complex number z ,
 $z + 0 = 0 + z = z$.
 - To every complex number $z = a + ib$, there exists another complex number $-z = -a + i(-b)$ called the additive inverse of z .
 $z + (-z) = (-z) + z = 0$
- Difference of two complex numbers:** Given any two complex numbers If $z_1 = a + ib$ and $z_2 = c + id$ the difference $z_1 - z_2$ is given by
 $z_1 - z_2 = z_1 + (-z_2) = (a - c) + i(b - d)$.
- Multiplication of two complex numbers** Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers. Then, the product $z_1 z_2$ is defined as follows:
 $z_1 z_2 = (ac - bd) + i(ad + bc)$
- Properties of multiplication of complex numbers:** Product of two complex numbers is a complex number; the product $z_1 z_2$ is a complex number for all complex numbers z_1 and z_2 .
 - Product of complex numbers is commutative, i.e. for any two complex numbers z_1 and z_2 , $z_1 z_2 = z_2 z_1$
 - Product of complex numbers is associative, i.e. for any three complex numbers z_1, z_2, z_3 , $(z_1 z_2) z_3 = z_1 (z_2 z_3)$.

- iii. There exists a complex number $1 + i0$ (denoted as 1), called the multiplicative identity such that $z \cdot 1 = z$ for every complex number z .
- iv. For every non-zero complex number, $z = a + ib$ or $a + bi$ ($a \neq 0$, $b \neq 0$), there is a complex number $\frac{a}{a^2 + b^2} + i \frac{-b}{a^2 + b^2}$ called the multiplicative inverse of z such that

$$z \times \frac{1}{z} = 1$$

- v. distributive law: For any three complex numbers z_1, z_2, z_3 ,

$$a. z_1(z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$$

$$b. (z_1 + z_2) z_3 = z_1 \cdot z_3 + z_2 \cdot z_3$$

7. **Division of two complex numbers:** Given any two complex numbers $z_1 = a + ib$ and $z_2 = c + id$, where $z_2 \neq 0$, the quotient $\frac{z_1}{z_2}$ is defined by

$$\frac{z_1}{z_2} = z_1 \cdot \frac{1}{z_2} = \frac{ac + bd}{c^2 + d^2} + i \frac{bc - ad}{c^2 + d^2}.$$

8. Identities for complex numbers

- i. $(z_1 + z_2)^2 = z_1^2 + z_2^2 + 2z_1 \cdot z_2$, for all complex numbers z_1 and z_2 .
- ii. $(z_1 - z_2)^2 = z_1^2 - 2z_1 z_2 + z_2^2$
- iii. $(z_1 + z_2)^3 = z_1^3 + 3z_1^2 z_2 + 3z_1 z_2^2 + z_2^3$
- iv. $(z_1 - z_2)^3 = z_1^3 - 3z_1^2 z_2 + 3z_1 z_2^2 - z_2^3$
- v. $z_1^2 - z_2^2 = (z_1 + z_2)(z_1 - z_2)$

9. Properties of modulus and conjugate of complex numbers

For any two complex numbers z_1 and z_2 ,

- i. $|z_1 z_2| = |z_1||z_2|$
- ii. $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$ provided $|z_2| \neq 0$
- iii. $\overline{z_1 z_2} = \overline{z_1} \overline{z_2}$
- iv. $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$
- v. $\overline{\left(\frac{z_1}{z_2} \right)} = \frac{\overline{z_1}}{\overline{z_2}}$ provided $z_2 \neq 0$
- vi. $\overline{(\overline{z})} = z$
- vii. $z + \overline{z} = 2\operatorname{Re}(z)$
- viii. $z - \overline{z} = 2i\operatorname{Im}(z)$
- ix. $z = \overline{z} \Leftrightarrow z$ is purely real
- x. $z + \overline{z} = 0 \Rightarrow z$ is purely imaginary
- xi. $z\overline{z} = [\operatorname{Re}(z)]^2 + [\operatorname{Im}(z)]^2$

10. The order of a relation is not defined in complex numbers. Hence there is no meaning in $z_1 > z$.
11. Two complex numbers z_1 and z_2 are equal iff $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$.
12. The sum and product of two complex numbers are real if and only if they are conjugate of each other.
13. For any integer k , $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$. $\sqrt{a} \times \sqrt{b} \neq \sqrt{ab}$ when $a < 0$ and $b < 0$.
14. The polar form of the complex number $z = x + iy$ is $r(\cos \theta + i \sin \theta)$, where r is the modulus of z and θ is known as the argument of z .
15. For a quadratic equation $ax^2 + bx + c = 0$ with real coefficients a , b and c and $a \neq 0$. If the discriminant $D = b^2 - 4ac \geq 0$, then the equation has two real roots given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b}{2a}.$$

16. Roots of the quadratic equation $ax^2 + bx + c = 0$, where a , b and $c \in \mathbb{R}$, $a \neq 0$, when discriminant $b^2 - 4ac < 0$, are imaginary given by

$$x = \frac{-b \pm \sqrt{4ac - b^2}i}{2a}.$$

17. Complex roots occur in pairs.
18. If a , b and c are rational numbers and $b^2 - 4ac$ is positive and a perfect square, then $\sqrt{b^2 - 4ac}$

is a rational number and hence the roots are rational and unequal.

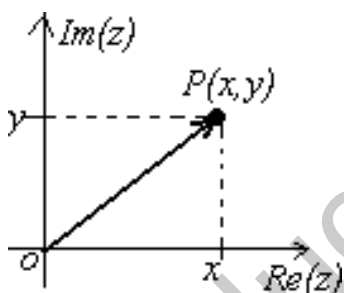
19. If $b^2 - 4ac = 0$, then the roots of the quadratic equation are real and equal.

20. If $b^2 - 4ac = 0$ but it is not a perfect square, then the roots of the quadratic equation are irrational and unequal.

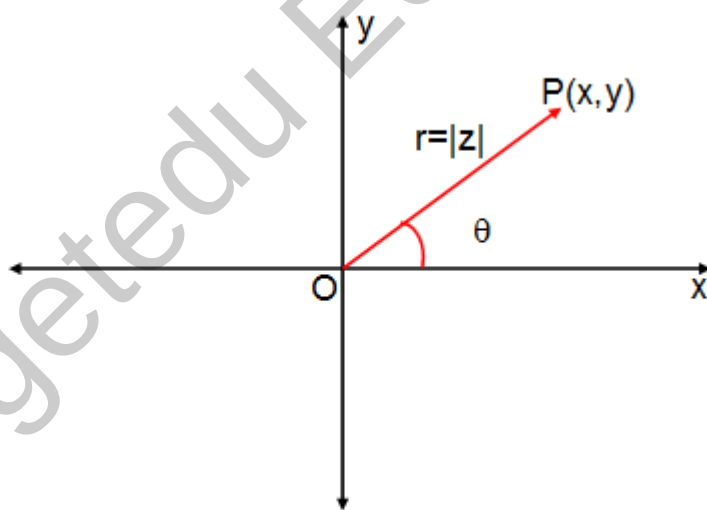
21. Irrational roots occur in pairs.

22. A polynomial equation of n degree has n roots. These n roots could be real or complex.

23. Complex numbers are represented in the Argand plane with X-axis being real and Y-axis being imaginary.

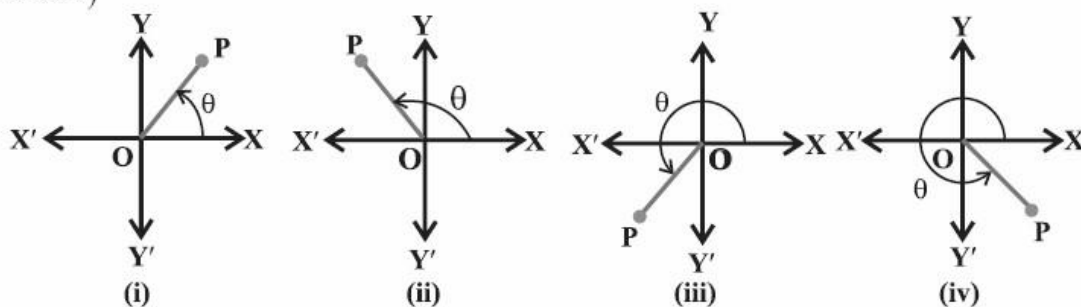
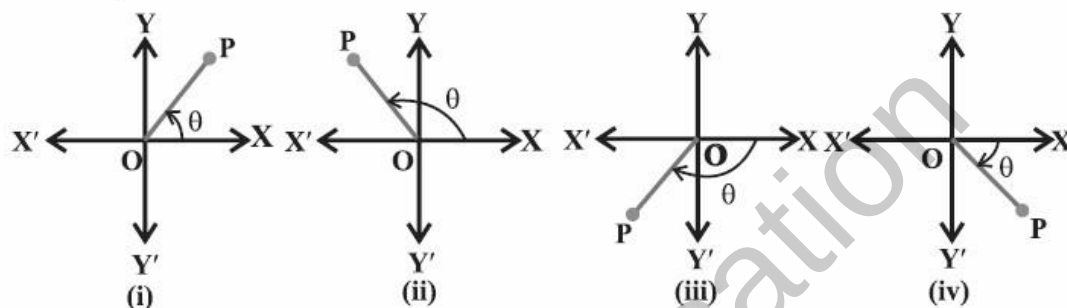


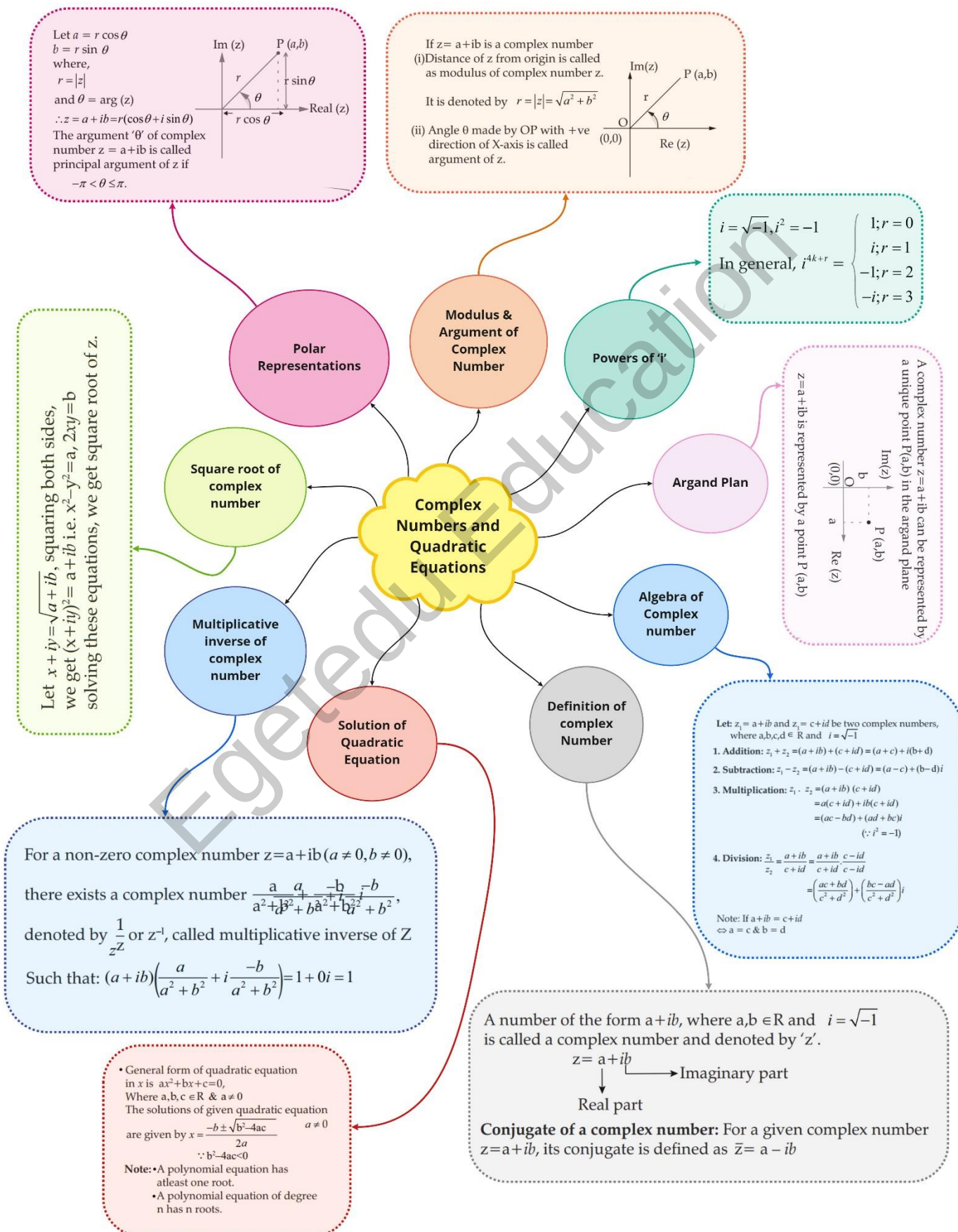
24. Representation of complex number $z = x + iy$ in the Argand plane.



25. Multiplication of a complex number by i results in rotating the vector joining the origin to the point representing z through a right angle.

26. Argument θ of the complex number z can take any value in the interval $[0, 2\pi)$. Different orientations of z are as follows

$(0 \leq \theta < 2\pi)$  $(-\pi < \theta \leq \pi)$ 



Important Questions

Multiple Choice questions-

Question 1. Let z_1 and z_2 be two roots of the equation $z^2 + az + b = 0$, z being complex. Further assume that the origin, z_1 and z_2 form an equilateral triangle. Then

- (a) $a^2 = b$
- (b) $a^2 = 2b$
- (c) $a^2 = 3b$
- (d) $a^2 = 4b$

Question 2. The value of i^i is

- (a) 0
- (b) $e^{-\pi}$
- (c) $2e^{-\pi/2}$
- (d) $e^{-\pi/2}$

Question 3. The value of $\sqrt{-25} + 3\sqrt{-4} + 2\sqrt{-9}$ is

- (a) $13i$
- (b) $-13i$
- (c) $17i$
- (d) $-17i$

So, $\sqrt{-25} + 3\sqrt{-4} + 2\sqrt{-9} = 17i$

Question 4. If the cube roots of unity are 1, ω and ω^2 , then the value of $(1 + \omega / \omega^2)^3$ is

- (a) 1
- (b) -1
- (c) ω
- (d) ω^2

Question 5. If $\{(1+i)/(1-i)\}^n = 1$ then the least value of n is

- (a) 1
- (b) 2
- (c) 3
- (d) 4

Question 6. The value of $[i^{19} + (1/i)^{25}]^2$ is

- (a) -1
- (b) -2
- (c) -3
- (d) -4

Question 7. If z and w be two complex numbers such that $|z| \leq 1$, $|w| \leq 1$ and $|z + iw| = |z - iw| = 2$, then z equals $\{w$ is conjugate of $w\}$

- (a) 1 or i
- (b) i or $-i$
- (c) 1 or -1
- (d) i or -1

Question 8. The value of $\{-\sqrt{-1}\}^{4n+3}$, $n \in \mathbb{N}$ is

- (a) i
- (b) $-i$
- (c) 1
- (d) -1

Question 9. Find real θ such that $(3 + 2i \times \sin \theta)/(1 - 2i \times \sin \theta)$ is real

- (a) π
- (b) $n\pi$
- (c) $n\pi/2$
- (d) $2n\pi$

Question 10. If $i = \sqrt{-1}$ then $4 + 5(-1/2 + i\sqrt{3}/2)^{334} + 3(-1/2 + i\sqrt{3}/2)^{365}$ is equals to

- (a) $1 - i\sqrt{3}$
- (b) $-1 + i\sqrt{3}$
- (c) $i\sqrt{3}$
- (d) $-i\sqrt{3}$

Very Short Questions:

Evaluate i^{-39}

1. Solved the quadratic equation $x^2 + x \frac{1}{\sqrt{2}} = 0$
2. If $\left(\frac{1+i}{1-i}\right)^m = 1$, then find the least positive integral value of m .
3. Evaluate $(1+i)^4$

- Find the modulus of $\frac{1+i}{1-i} - \frac{1-i}{1+i}$
- Express in the form of $a + ib$. $(1+3i)^{-1}$
- Explain the fallacy in $-1 = i \cdot i = \sqrt{-1} \cdot \sqrt{-1} = \sqrt{-1(-1)} = \sqrt{1} = 1$.
- Find the conjugate of $\frac{1}{2-3i}$
- Find the conjugate of $-3i - 5$.
- Let $z_1 = 2 - i$, $z_2 = -2 + i$ Find $\operatorname{Re} \left(\frac{z_1 z_2}{z_1} \right)$

Short Questions:

- If $x + iy = \frac{a+ib}{a-ib}$ Prove that $x^2 + y^2 = 1$
- Find real θ such that $\frac{3+2i \sin \theta}{1-2i \sin \theta}$ is purely real.
- Find the modulus of $\frac{(1+i)(2+i)}{3+i}$
- If $|a + ib| = 1$ then Show that $\frac{1+b+ai}{1+b-ai} = b + ai$
- If $x - iy = \sqrt{\frac{a-ib}{c-id}}$ Prove that $(x^2 + y^2)^2 = \frac{a^2+b^2}{c^2+d^2}$

Long Questions:

- If $z = x + iy$ and $w = \frac{1-i^2}{z-i}$ Show that $|w| = 1 \Rightarrow z$ is purely real.
- Convert into polar form $\frac{-16}{1+i\sqrt{3}}$
- Find two numbers such that their sum is 6 and the product is 14.
- Convert into polar form $z = \frac{i-1}{\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}}$
- If α and β are different complex number with $|\beta| = 1$ Then find $\left| \frac{\beta-\alpha}{1-\alpha\beta} \right|$

Assertion Reason Questions:

- In each of the following questions, a statement of Assertion is given followed by a corresponding statement of Reason just below it. Of the statements, mark the correct answer as.

Assertion (A): If $i = \sqrt{-1}$, then $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$ and $i^{4k+3} = -i$.

Reason (R): $i^{4k} + i^{4k+1} + i^{4k+2} + i^{4k+3} = 1$.

- (i) Both assertion and reason are true and reason is the correct explanation of assertion.
 - (ii) Both assertion and reason are true but reason is not the correct explanation of assertion.
 - (iii) Assertion is true but reason is false.
 - (iv) Assertion is false but reason is true.
2. In each of the following questions, a statement of Assertion is given followed by a corresponding statement of Reason just below it. Of the statements, mark the correct answer as.

Assertion (A): Simplest form of i^{-35} is $-i$.

Reason (R) : Additive inverse of $(1 - i)$ is equal to $-1 + i$.

- (i) Both assertion and reason are true and reason is the correct explanation of assertion.
- (ii) Both assertion and reason are true but reason is not the correct explanation of assertion.
- (iii) Assertion is true but reason is false.
- (iv) Assertion is false but reason is true.

Answer Key:

MCQ

1. (c) $a^2 = 3b$
2. (d) $e^{-\pi/2}$
3. (c) $17i$
4. (b) -1
5. (d) 4
6. (d) -4
7. (c) 1 or -1
8. (a) i
9. (b) $n\pi$
10. (c) $i\sqrt{3}$

Very Short Answer:

- 1.

$$\begin{aligned}
 i^{-39} &= \frac{1}{i^{39}} = \frac{1}{(i^4)^9 \cdot i^3} \\
 &= \frac{1}{1 \times (-i)} \quad \left[\begin{array}{l} \because i^4 = 1 \\ i^3 = -i \end{array} \right] \\
 &= \frac{1}{-i} \times \frac{i}{i} \\
 &= \frac{i}{-i^2} = \frac{i}{-(-1)} = i \quad [\because i^2 = -1]
 \end{aligned}$$

2.

$$\begin{aligned}
 \frac{x^2}{1} + \frac{x}{1} + \frac{1}{\sqrt{2}} &= 0 \\
 \frac{\sqrt{2}x^2 + \sqrt{2}x + 1}{\sqrt{2}} &= \frac{0}{1} \\
 \sqrt{2}x^2 + \sqrt{2}x + 1 &= 0 \\
 x &= \frac{-b \pm \sqrt{D}}{2a} \\
 &= \frac{-\sqrt{2} \pm \sqrt{2 - 4\sqrt{2}}}{2 \times \sqrt{2}} \\
 &= \frac{-\sqrt{2} \pm \sqrt{2}\sqrt{1 - 2\sqrt{2}}}{2\sqrt{2}} \\
 &= \frac{-1 \pm \sqrt{2\sqrt{2} - 1}}{2} i
 \end{aligned}$$

3.

$$\begin{aligned}
 \left(\frac{1+i}{1-i} \right)^m &= 1 \\
 \left(\frac{1+i}{1-i} \times \frac{1+i}{1+i} \right)^m &= 1 \\
 \left(\frac{1+i^2+2i}{1-i^2} \right)^m &= 1 \\
 \left(\frac{1-1+2i}{2} \right)^m &= 1 \quad [\because i^2 = -1]
 \end{aligned}$$

$$\begin{aligned}
 i^m &= 1 \\
 m &= 4
 \end{aligned}$$

4.

$$\begin{aligned}
 (1+i)^4 &= \left[(1+i)^2\right]^2 \\
 &= (1+i^2+2i)^2 \\
 &= (1-1+2i)^2 \\
 &= (2i)^2 = 4i^2 \\
 &= 4(-1) = -4
 \end{aligned}$$

5.

$$\begin{aligned}
 \text{Let } z &= \frac{1+i}{1-i} - \frac{1-i}{1+i} \\
 &= \frac{(1+i)^2 - (1-i)^2}{(1-i)(1+i)} \\
 &= \frac{4i}{2} \\
 &= 2i \\
 z &= 0 + 2i \\
 |z| &= \sqrt{(0)^2 + (2)^2} \\
 &= 2
 \end{aligned}$$

6.

$$\begin{aligned}
 (1+3i)^{-1} &= \frac{1}{1+3i} \times \frac{1-3i}{1-3i} \\
 &= \frac{1-3i}{(1)^2 - (3i)^2} \\
 &= \frac{1-3i}{1-9i^2} \\
 &= \frac{1-3i}{1+9} \quad [i^2 = -1] \\
 &= \frac{1-3i}{10} \\
 &= \frac{1}{10} - \frac{3i}{10}
 \end{aligned}$$

7.

$$\begin{aligned}
 1 &= \sqrt{1} = \sqrt{(-1)(-1)} \text{ is okay but} \\
 \sqrt{(-1)(-1)} &= \sqrt{-1}\sqrt{-1} \text{ is wrong.}
 \end{aligned}$$

8.

$$\text{Let } z = \frac{1}{2-3i}$$

$$z = \frac{1}{2-3i} \times \frac{2+3i}{2+3i}$$

$$= \frac{2+3i}{(2)^2 - (3i)^2}$$

$$= \frac{2+3i}{4+9}$$

$$= \frac{2+3i}{13}$$

$$z = \frac{2}{13} + \frac{3}{13}i$$

$$\bar{z} = \frac{2}{13} - \frac{3}{13}i$$

9. Let $z = 3i - 5$

$$\bar{z} = 3i - 5$$

10. $z_1 z_2 = (2-i)(-2+i)$

$$= -4 + 2i + 2i - i^2$$

$$= -4 + 4i + 1$$

$$= 4i - 3$$

$$\bar{z}_1 = 2+i$$

$$\frac{z_1 z_2}{\bar{z}_1} = \frac{4i-3}{2+i} \times \frac{2-i}{2-i}$$

$$= \frac{8i - 6 - 4i^2 + 3i}{4 - i^2}$$

$$= \frac{11i - 2}{5}$$

$$\frac{z_1 z_2}{z_1} = \frac{11}{5}i - \frac{2}{5}$$

$$\operatorname{Re}\left(\frac{z_1 z_2}{\bar{z}_1}\right) = -\frac{2}{5}$$

Short Answer:

1.

$$x+iy = \frac{a+ib}{a-ib} \quad (\text{i}) \quad (\text{Given})$$

taking conjugate both side

$$x - iy = \frac{a - ib}{a + ib} \quad (\text{ii})$$

$$(\text{i}) \times (\text{ii})$$

$$(x + iy)(x - iy) = \left(\frac{a + ib}{a - ib} \right) \times \left(\frac{a - ib}{a + ib} \right)$$

$$(x)^2 - (iy)^2 = 1$$

$$x^2 + y^2 = 1$$

$$[i^2 = -1]$$

2.

$$\frac{3 + 2i \sin \theta}{1 - 2i \sin \theta} = \frac{3 + 2i \sin \theta}{1 - 2i \sin \theta} \times \frac{1 + 2i \sin \theta}{1 + 2i \sin \theta}$$

$$= \frac{3 + 6i \sin \theta + 2i \sin \theta - 4 \sin^2 \theta}{1 + 4 \sin^2 \theta}$$

$$= \frac{3 - 4 \sin^2 \theta}{1 + 4 \sin^2 \theta} + \frac{8i \sin \theta}{1 + 4 \sin^2 \theta}$$

For purely real

$$\text{Im}(z) = 0$$

$$\frac{8 \sin \theta}{1 + 4 \sin^2 \theta} = 0$$

$$\sin \theta = 0$$

$$\theta = n\pi$$

3.

$$\left| \frac{(1+i)(2+i)}{3+i} \right| = \frac{|(1+i)| |2+i|}{|3+i|}$$

$$= \frac{(\sqrt{1^2+1^2})(\sqrt{4+1})}{\sqrt{(3)^2+(1)^2}}$$

$$= \frac{(\sqrt{2})(\sqrt{5})}{\sqrt{10}}$$

$$= \frac{\sqrt{2} \times \sqrt{5}}{\sqrt{2} \times \sqrt{5}}$$

$$= 1$$

4.

$$|a+ib|=1$$

$$\sqrt{a^2+b^2}=1$$

$$a^2+b^2=1$$

$$\frac{1+b+ai}{1+b-ai} = \frac{(1+b)+ai}{(1+b)-ai} \times \frac{(1+b)+ai}{(1+b)+ai}$$

$$= \frac{(1+b)^2 + (ai)^2 + 2(1+b)(ai)}{(1+b)^2 - (ai)^2}$$

$$= \frac{1+b^2+2b-a^2+2ai+2abi}{1+b^2+2a-a^2}$$

$$= \frac{(a^2+b^2)+b^2+2b-a^2+2ai+2abi}{(a^2+b^2)+b^2+2b-a^2}$$

$$= \frac{2b^2+2b+2ai+2abi}{2b^2+2b}$$

$$= \frac{b^2+b+ai+abi}{b^2+b}$$

$$= \frac{b(b+1)+ai(b+1)}{b(b+1)}$$

$$= b+ai$$

5.

$$x-iy = \sqrt{\frac{a-ib}{c-id}} \quad (1) \text{ (Given)}$$

Taking conjugate both side

$$x+iy = \sqrt{\frac{a+ib}{c+id}} \quad (ii)$$

(i) $\times$ (ii)

$$(x-iy) \times (x+iy) = \sqrt{\frac{a-ib}{c-id}} \times \sqrt{\frac{a+ib}{c+id}}$$

$$(x)^2 - (iy)^2 = \sqrt{\frac{(a)^2 - (ib)^2}{(c)^2 - (id)^2}}$$

$$x^2 + y^2 = \sqrt{\frac{a^2 + b^2}{c^2 + d^2}}$$

squaring both side

$$(x^2 + y^2)^2 = \frac{a^2 + b^2}{c^2 + d^2}$$

Long Answer:

1.

$$w = \frac{1-iz}{z-i}$$

$$= \frac{1-i(x+iy)}{x+iy-i}$$

$$= \frac{1-ix-i^2y}{x+i(y-1)}$$

$$= \frac{(1+y)-ix}{x+i(y-1)}$$

$$\therefore |w|=1$$

$$\Rightarrow \left| \frac{(1+y)-ix}{x+i(y-1)} \right| = 1$$

$$\frac{|(1+y)-ix|}{|x+i(y-1)|} = 1$$

$$\frac{\sqrt{(1+y)^2 + (-x)^2}}{\sqrt{x^2 + (y-1)^2}} = 1$$

$$1+y^2+2y+x^2 = x^2+y^2+1-2y$$

$$4y = 0$$

$$y = 0$$

$$\therefore z = x + i$$

is purely real

2.

$$\frac{-16}{1+i\sqrt{3}} = \frac{-16}{1+i\sqrt{3}} \times \frac{1-i\sqrt{3}}{1-i\sqrt{3}}$$

$$= \frac{-16(1-i\sqrt{3})}{(1)^2 - (i\sqrt{3})^2}$$

$$= \frac{-16(1-i\sqrt{3})}{1+3}$$

$$= -4(1-i\sqrt{3})$$

$$z = -4 + i4\sqrt{3}$$

$$r = |z| = \sqrt{(-4)^2 + (4\sqrt{3})^2}$$

$$= \sqrt{16+48}$$

$$= \sqrt{64}$$

$$= 8$$

Let α be the acute $\angle S$

$$\tan \alpha = \left| \frac{\cancel{4}\sqrt{3}}{\cancel{4}} \right|$$

$$\tan \alpha = \tan \frac{\pi}{3}$$

$$\alpha = \pi/3$$

Since $\operatorname{Re}(z) < 0$, and $\operatorname{Im}(z) > 0$

$$\theta = \pi - \alpha$$

$$= \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

$$z = 8 \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right)$$

3.

Let x and y be the no.

$$x + y = 6$$

$$xy = 14$$

$$x^2 - 6x + 14 = 0$$

$$D = -20$$

$$x = \frac{-(-6) \pm \sqrt{-20}}{2 \times 1}$$

$$= \frac{6 \pm 2\sqrt{5}i}{2}$$

$$= 3 \pm \sqrt{5}i$$

$$x = 3 + \sqrt{5}i$$

$$y = 6 - (3 + \sqrt{5}i)$$

$$= 3 - \sqrt{5}i$$

$$\text{when } x = 3 - \sqrt{5}i$$

$$y = 6 - (3 - \sqrt{5}i)$$

$$= 3 + \sqrt{5}i$$

4.

$$z = \frac{i-1}{\frac{1}{2} + \frac{\sqrt{3}}{2}i}$$

$$= \frac{2(i-1)}{1+\sqrt{3}i} \times \frac{1-\sqrt{3}i}{1-\sqrt{3}i}$$

$$z = \frac{\sqrt{3}-1}{2} + \frac{\sqrt{3}+1}{2}i$$

$$r = |z| = \left(\frac{\sqrt{3}-1}{2}\right)^2 + \left(\frac{\sqrt{3}+1}{2}\right)^2$$

$$r = 2$$

Let α be the acute $\angle$ s

$$\tan \alpha = \left| \frac{\frac{\sqrt{3}+1}{2}}{\frac{\sqrt{3}-1}{2}} \right|$$

$$= \left| \frac{\sqrt{3}\left(1+\frac{1}{\sqrt{3}}\right)}{\sqrt{3}\left(1-\frac{1}{\sqrt{3}}\right)} \right|$$

$$= \left| \frac{\tan \frac{\pi}{4} + \tan \frac{\pi}{6}}{1 - \tan \frac{\pi}{4} \tan \frac{\pi}{6}} \right|$$

$$\tan \alpha = \left| \tan \left(\frac{\pi}{4} + \frac{\pi}{6} \right) \right|$$

$$\alpha = \frac{\pi}{4} + \frac{\pi}{6} = \frac{5\pi}{12}$$

$$z = 2 \left(\cos \frac{5\pi}{12} + i \sin \frac{5\pi}{12} \right)$$

5.

$$\left| \frac{\beta - \alpha}{1 - \overline{\alpha}\beta} \right|^2 = \left(\frac{\beta - \alpha}{1 - \overline{\alpha}\beta} \right) \left(\frac{\overline{\beta - \alpha}}{1 - \overline{\alpha}\beta} \right) \quad [\because |z|^2 = z\overline{z}]$$

$$\begin{aligned}
&= \left(\frac{\beta - \alpha}{1 - \bar{\alpha}\beta} \right) \left(\frac{\bar{\beta} - \bar{\alpha}}{1 - \alpha\bar{\beta}} \right) \\
&= \left(\frac{\beta\bar{\beta} - \beta\bar{\alpha} - \alpha\bar{\beta} + \alpha\bar{\alpha}}{1 - \alpha\bar{\beta} - \bar{\alpha}\beta + \alpha\bar{\alpha}\beta\bar{\beta}} \right) \\
&= \left(\frac{|\beta|^2 - \beta\bar{\alpha} - \alpha\bar{\beta} + |\alpha|^2}{1 - \alpha\bar{\beta} - \bar{\alpha}\beta + |\alpha|^2 |\beta|^2} \right) \\
&= \left(\frac{1 - \beta\bar{\alpha} - \alpha\bar{\beta} + |\alpha|^2}{1 - \alpha\bar{\beta} - \bar{\alpha}\beta + |\alpha|^2} \right) \quad [\because |\beta|=1] \\
&= 1 \\
&\left| \frac{\beta - \alpha}{1 - \bar{\alpha}\beta} \right| = \sqrt{1} \\
&\left| \frac{\beta - \alpha}{1 - \bar{\alpha}\beta} \right| = 1
\end{aligned}$$

Assertion Reason Answer:

1. (iii) Assertion is true but reason is false.
2. (iv) Assertion is false but reason is true.